References

[1] Zimmerman, N. H., "Elementary static aerodynamics adds significance and scope in flutter analysis," Symposium Proc. High Speed Flight, ONR-ACR-62, Vol. 1, Los Angeles, Calif. (April 1961).

[2] Bisplinghoff, R. L. and Ashley, H., *Principles of Aeroelasticity* (John Wiley and Sons, New York, 1962), pp. 269–270.

[3] De Young, J. and Harper, C. W., "Theoretical symmetric span loading at subsonic speeds for wings having arbitrary plan form," NACA TR 921 (1948).

[4] De Young, J., "Theoretical antisymmetric span loading for wings of arbitrary plane form at subsonic speeds," NACA TR 1056 (1951).

[5] Bodewig, E., *Matrix Calculus* (Interscience Publishers, New York, 1959), 2nd ed., pp. 285–289.

[6] Goland, M., "The flutter of a uniform cantilever wing," J. Appl. Mech. 12A, 197 (1945.).

Elastic Stability of Near-Perfect Shallow Spherical Shells

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Seventeen accurately machined spherical segments were collapsed under external hydrostatic pressure to determine the elastic buckling strength of near-perfect shallow spherical shells with clamped edges. Whereas previous experiments of less perfect shells recorded in the literature showed a complete lack of repeatability, the present tests follow a very definite pattern. Therefore, these tests demonstrate the critical effect of initial imperfections on collapse strength. The theories for symmetric buckling developed by Budiansky, Weinitschke, and Thurston, when combined with the theory for nonsymmetric buckling recently developed by Huang, may be used to adequately predict the behavior of the shells tested. Thus, for the first time there is good agreement between experiment and theory throughout the range of shallow spherical shells.

THE nonlinear theory for the elastic buckling strength of near-perfect shallow spherical segments with clamped edges under external hydrostatic pressure was first investigated by Feodosiev[1] in 1946. Since that time, the problem has received considerable attention. Solutions for the elastic axisymmetric buckling of initially perfect shallow spherical segments independently developed by Budiansky,[2] Weinitschke,[3] and Thurston[4] agree well with each other, but do not agree well with existing experimental data.[5-8] This disagreement is normally attributed to the fact that their theories neglected the presence of initial imperfections in the test specimens (which were formed from flat plate) and the influence of the nonsymmetric buckling mode. Both Chen[9] and Budiansky[2] have shown that the presence of initial imperfections lowers the theoretical elastic axisymmetric buckling pressure of shallow segments. However, the reduction in buckling pressure calculated for measured initial imperfections is not nearly enough to produce even fair agreement with the experimental data.[2]

Weinitschke[10] developed a theory for the nonsymmetric buckling of spherical segments which is in fair agreement with the experimental data existing prior to these present tests. Concurrent with the conduct of the present series of tests, Huang[11] developed a nonsymmetric buckling theory which was not supported by the earlier data.

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The present series of tests of machined spherical segments with clamped edges was designed to investigate the collapse strength of shells which more closely fulfill the assumptions of existing theory than was accomplished by previous experiments.

Description of Models

Seventeen shallow spherical segments with clamped edges, designated Models SS-57 through SS-73, were machined from 7075-T6 aluminum bar stock with a nominal yield strength of 80,000 psi. Young's modulus E, as determined by optical strain gage measurements, was 10.8×10^6 psi. A Poisson's ratio ν of 0.3 was assumed. The model dimensions are given in Table 1.

Departures from sphericity, variations in thickness, and residual stresses were minimized by careful selection and conduct of the machining processes. The measured variation in local inside radii for each model was less than 0.0002 in. and normally less than 0.0001 in. The measured variations in shell thickness, which are shown in Table 1, were normally less than 1% of the shell thickness. The geometric stability of the models during the final machining processes demonstrates that negligible residual stresses were present in the models after machining.

Test Procedure and Results

Each model was tested under external hydrostatic pressure. Pressure was applied in increments, and each new pressure level was held at least 1 min. The final pressure increment was less than 1% of the collapse pressure for each model. An effort was made to minimize any pressure surge when applying load.

Experimental collapse pressures are given in Table 1. Some of the models collapsed in a nonsymmetric mode, and others apparently collapsed in a symmetric mode.

Discussion

The collapse pressures of these models are compared in Fig. 1 with test results recorded in the literature and with nonlinear symmetric and nonsymmetric theory. The ordinate is the ratio of the experimental collapse pressure p_{exp} to the classical small deflection buckling pressure p_1 as developed by Zoelly and presented by Timoshenko.[11] The abscissa is the nondimensional geometric parameter θ defined as

$$\theta = \left[\frac{3}{4}(1 - \nu^2)\right]^{1/4} \frac{C}{(Rh)^{1/2}} = \frac{0.91C}{(Rh)^{1/2}} \text{ for } \nu = 0.3 \quad (1)$$

where R is the radius to the midsurface of the shell, h is the

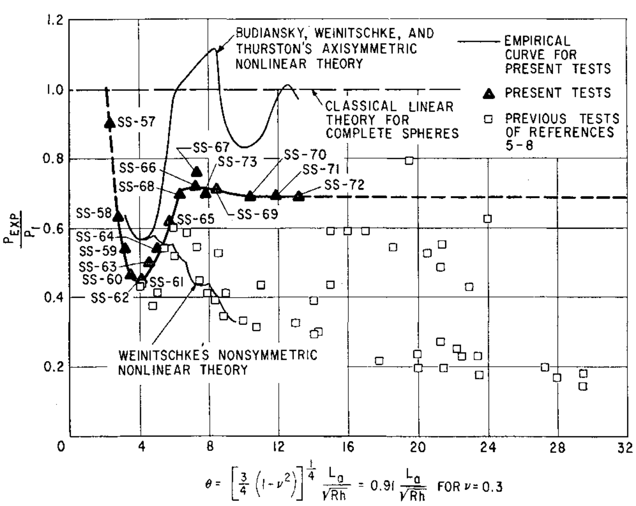
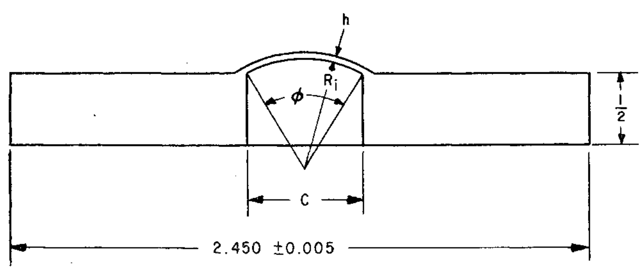


Fig. 1. Experimental elastic buckling data for shallow spherical shells with clamped edges.

Table 1 Dimensions and experimental collapse pressures

Model	Measured wall thickness h, in.	Measured variation in h, in.	Inside radius R_i, in.	Inner chord C, in.	Included angle ϕ, deg	Experimental collapse pressure P_{exp}, psi
SS-57	0.0394	+0.0000 −0.0001	3.00	0.86	16.5	2050
SS-58	0.0346	+0.0001 −0.0000	3.00	0.96	18.5	1080
SS-59	0.0298	+0.0000 −0.0001	3.00	1.04	20.0	690
SS-60	0.0301	+0.0000 −0.0001	3.00	1.19	22.8	605
SS-61	0.0299	±0.0001	3.00	1.33	25.7	580
SS-62	0.0300	+0.0000 −0.0001	3.00	1.33	25.7	570
SS-63	0.0200	+0.0000 −0.0001	3.00	1.32	23.4	290
SS-64	0.0200	+0.0000 −0.0001	3.00	1.33	25.7	312
SS-65	0.0147	+0.0000 −0.0001	3.00	1.26	24.3	193
SS-66	0.0147	±0.0003	3.00	1.67	32.4	225
SS-67	0.0141	+0.0002 −0.0004	3.00	1.67	32.4	218
SS-68	0.0100	+0.0001 −0.0000	2.00	0.97	28.1	226
SS-69	0.0096	±0.0002	2.00	1.31	38.1	213
SS-70	0.0107	±0.0001	2.00	1.57	46.0	257
SS-71	0.0105	±0.0001	2.00	1.82	54.0	249
SS-72	0.0105	±0.0001	2.00	2.00	60.0	246
SS-73	0.0104	+0.0000 −0.0001	2.00	1.20	35.0	245



shell thickness, and C is the unsupported chord length of spherical shell.

Whereas previous experiments recorded in the literature showed a complete lack of repeatability, the present results follow a very definite pattern. Since the primary difference between these and earlier specimens was the magnitude of initial departures from sphericity, these results demonstrate the detrimental effects on collapse strength of initial imperfections. These results also demonstrate that a short clamped segment can be weaker than a longer clamped segment. Although this phenomenon has been implied by existing theoretical studies, it found no support in the earlier experiments.

Because of the definite pattern established by these tests, a comparison can be made between the empirical curve and the available theories for a wide range of θ. For short segments associated with values of θ less than about 5.5, the empirical curve arbitrarily drawn through the experimental points in Fig. 1 has the same general shape as the curve representing the theory for symmetric buckling.[2–4] However, as the shells become longer or deeper, the empirical curve departs from the theoretical symmetric buckling curve and is in good agreement with the nonsymmetric theory developed by Huang. Hence, as indicated by the experimental results and the theory, it is reasonable to assume that the mode of collapse becomes nonsymmetric at θ values of about 5.5. Actually, the deformed surfaces of the three longest shells after collapse were nonsymmetric, whereas the others were symmetric. This does not mean that all of the 14 shallower segments failed in the symmetric mode. Previous investigators have found that failures which appear to be symmetric after collapse may have been initiated in the nonsymmetric mode.[5,8] Although Weinitschke's nonsymmetric theory[10] is in fair agreement with early experiments, it is not supported by the present results.

The empirical curve is of the same general shape as the theoretical curves, but it lies about 10–20% below the theoretical curve. This is consistent with recent observations of the elastic buckling strength of near-perfect, deep-spherical shells.[13] These tests demonstrated that for small, almost unmeasurable imperfections, the buckling pressure of near-perfect spheres may vary from a value approaching the classical buckling pressure to 70% or less of the classical pressure. Thus, it is not surprising that the present results are somewhat below the theoretical curves.

Based on the results of these tests, several important conclusions can be drawn concerning the strength of shallow, and of deep spherical shells. First, the buckling strength increases rapidly for descending values of θ below about 2.2–2.5. Thus, if stiffening systems are installed in an attempt to improve structural efficiency, it is apparent that they must be placed at relatively close intervals in order to realize an increase in collapse strength. Placing stiffeners at spacings greater than the arc length corresponding to a θ of about 2.5 will not increase the local buckling strength of the shell and may possibly weaken it. Another less obvious conclusion obtained from these tests is related to initial departures from sphericity or local "flat spots." If a spherical shell contains a "flat spot" covering an arc length associated with a θ of about 2.2 or greater, the collapse strength of this shell would appear to be relatively independent of the nominal radius of the shell. Based on the local curvature of the "flat spot" rather than on the nominal curvature which is commonly used, a new upper bound may be determined for the collapse strength of an initially imperfect spherical shell. This observation is the basis of an analysis developed in Ref. 14 which adequately predicts both the elastic and inelastic strength of 36 machined models with local "flat spots" covering a relatively wide range of θ.

References

[1] Feodosiev, V. I., "Calculation of thin clicking membranes," Prikl. Mat. Mekh., **X**, 295 (1946).

[2] Budiansky, B., "Buckling of clamped shallow spherical shells," *Proceedings of the IUTAM Symposium of the Theory of Thin Elastic Shells* (North Holland Publishing Co., Amsterdam, 1960), p. 64.

[3] Weinitschke, H., "On the stability problem for shallow spherical shells," J. Math. Phys. **38**, 209 (January 1960).

[4] Thurston, G. A., "A numerical solution of the non-linear equations for axisymmetric bending of shallow spherical shells," J. Appl. Mech. **28**, 557 (December 1961).

[5] Kloppel, K. and Jungbluth, O., "Beitrag zum Durchschlagsproblem dunwandieger Kugelschalen (Versuche und Bemessungsformeln)," Der Stahlbau, **22**, 121 (1953).

[6] Tsien, H. S., "A theory for the buckling of thin shells," J. Aeronaut. Sci. **9**, 373 (August 1942).

[7] Kaplan, A. and Fung, Y. C., "A non-linear theory of bending and buckling of thin elastic shallow spherical shells," NACA TN 3212 (August 1954).

[8] Homewood, R. H., Brine, A. C., and Johnson, A. E., Jr., "Experimental investigation of the buckling instability of monocoque shells," Exptl. Mech. **1**, 88 (March 1961).

[9] Chen, W. L., "Effect of geometrical imperfections on the elastic buckling of shallow spherical shells," Sc.D. Thesis, Mass. Inst. Tech. (January 1959).

[10] Weinitschke, H., "Asymmetric buckling of clamped shallow spherical shells," NASA TN D-1510, 481 (December 1962).

[11] Huang, N. C., "Unsymmetrical buckling of thin shallow spherical shells," Tech. Rept. 15, Harvard Univ. (March 1963).

[12] Timoshenko, S., *Theory of Elastic Stability* (McGraw-Hill Book Co., Inc., New York, 1936), p. 491.

[13] Krenzke, M. A., "The elastic buckling strength of near-perfect deep spherical shells with ideal boundaries," David Taylor Model Basin Rept. 1713 (July 1963).

[14] Krenzke, M. A. and Kiernan, T. J., "The effect of initial imperfections on the collapse strength of deep spherical shells," David Taylor Model Basin Rept. 1757 (in preparation).

Some Observations on the Nonlinear Vibration of Thin Cylindrical Shells

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RECENT works[1, 2] on the nonlinear vibration of cylindrical shells indicated that 1) the nonlinearity was of the "hardening" type, and 2) in some cases the problem was rather strongly nonlinear.

Suspecting that these phenomena would be readily detectable in the laboratory, the author performed a few experiments in which shells were vibrated at amplitudes of three to four wall thicknesses. The experimental results indicated that 1) the nonlinearity was of the "softening" type, and, 2) for the shells that were tested, the vibrations were only slightly nonlinear.

This led to a re-examination of the analysis. Using the functions for the displacement normal to the shell w and the stress function F from Ref. 1, it appears impossible to satisfy the constraint that the midplane circumferential displacement v be continuous and single-valued. That is, using a common notation (see Fig. 1), one cannot satisfy

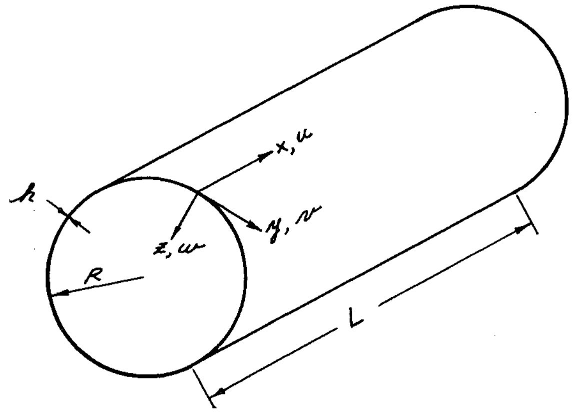

Fig. 1 Shell geometry and co-ordinate system.

$$\oint \frac{\partial v}{\partial y}\, dy = 0 \qquad (1)$$

for all x with w and F as given by Chu.

Reissner, in his earlier work,[3] isolated a segment or "lobe" of a cylindrical shell and examined its nonlinear behavior. He implied that his results were applicable to complete cylindrical shells, but the continuity condition (1) was not satisfied. It is not surprising that his results are similar to those of Cummings[4] who studied curved panels. It may be noted in passing that nonlinear static stability analyses have made use of (1) since the 1941 paper of von Karman and Tsien.[5]

After some preliminary investigations, it was found that a deflection function of the form

$$w(x,y,t) = A(t) \sin\frac{m\pi x}{L} \sin\frac{ny}{R} + \frac{n^2A^2(t)}{4R} \sin^2\frac{m\pi x}{L} \qquad (2)$$

together with the associated stress function, will satisfy the constraint (1) on v and still give zero normal displacement at the ends of the shell (at $x = 0$, $x = L$). With this combination of w and F, results were obtained that are in qualitative agreement with the experiments. The derivation was performed first using Galerkin's procedure with a weighting function given by $\partial w/\partial A$, and then the calculations were checked by employing the energy method. As in the previous papers, the author utilized the well-known shallow shell equations.

Shortly thereafter, the paper by Nowinski[2] appeared, confirming Chu's results and satisfying condition (1). It is of interest to note that Chu satisfies $w = 0$ at the ends of the shell but not the constraint (1) on v, whereas Nowinski satisfies the latter condition but not the former. Yet, they arrived at virtually identical results (for the isotropic case). This seems somewhat surprising at first, but possibly the answer lies in the fact that most approaches to date tacitly ignore the complementary solutions of the compatability equation

$$\frac{\nabla^4 F}{Eh} = -\frac{w_{xx}}{R} + [w_{xy}^2 - w_{xx}w_{yy}] \qquad (3)$$

where

$$w_{xx} = \frac{\partial^2 w}{\partial x^2}, \text{etc.}$$

In fact, by adding a nonzero solution of $\nabla^4 F = 0$ to the stress function given by Chu, it is possible to satisfy (1) and still obtain solutions quite similar to his original results. It also

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